## Chapter review 5

**1** $\mathbf{A} = \begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$

$\det \mathbf{A} = 1 \times 1 - 2 \times -3$
$= 7$

$\mathbf{A}^{-1} = \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}$

$\mathbf{B} = \mathbf{A}^{-1}\mathbf{AB}$

$= \frac{1}{7}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 & 9 \\ 1 & 9 & 4 \end{pmatrix}$

$= \frac{1}{7}\begin{pmatrix} 7 & 28 & 21 \\ -7 & 7 & -14 \end{pmatrix}$

$= \begin{pmatrix} 1 & 4 & 3 \\ -1 & 1 & -2 \end{pmatrix}$

**2 a** $\mathbf{A} = \begin{pmatrix} a & b \\ 2a & 3b \end{pmatrix}$

$\det \mathbf{A} = 3ab - 2ab$
$= ab$

$\mathbf{A}^{-1} = \frac{1}{ab}\begin{pmatrix} 3b & -b \\ -2a & a \end{pmatrix}$

$= \begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ -\frac{2}{b} & \frac{1}{b} \end{pmatrix}$

**b** $\mathbf{XA} = \mathbf{Y} = \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}$

$\mathbf{X} = \mathbf{XAA}^{-1}$

$= \begin{pmatrix} a & 2b \\ 2a & b \end{pmatrix}\begin{pmatrix} \frac{3}{a} & -\frac{1}{a} \\ -\frac{2}{b} & \frac{1}{b} \end{pmatrix}$

$= \begin{pmatrix} -1 & 1 \\ 4 & -1 \end{pmatrix}$

**3 a** $\mathbf{A} = \begin{pmatrix} k & -2 \\ -4 & k \end{pmatrix}$

$\det \mathbf{A} = k^2 - 8$

If the matrix has an inverse then $\det \mathbf{A} \neq 0 \Rightarrow k^2 - 8 \neq 0 \Rightarrow k \neq \pm 2\sqrt{2}$

**b** $\mathbf{A}^{-1} = \frac{1}{k^2 - 8}\begin{pmatrix} k & 2 \\ 4 & k \end{pmatrix}$

**4 a** $\mathbf{M} = \begin{pmatrix} 2 & -m \\ m & -1 \end{pmatrix}$

det $\mathbf{M} = -2 + m^2$

If $\mathbf{M}$ is singular then det $\mathbf{A} = 0 \Rightarrow m^2 - 2 = 0 \Rightarrow m = \pm\sqrt{2}$

**b** $\mathbf{M}^{-1} = \dfrac{1}{m^2 - 2}\begin{pmatrix} -1 & m \\ -m & 2 \end{pmatrix}$

**5 a** Given $\mathbf{AB} = \mathbf{BA}$

and $\mathbf{ABA} = \mathbf{B}$

$\Rightarrow \quad \mathbf{A}(\mathbf{AB}) = \mathbf{B}$

$\Rightarrow \quad \mathbf{A}^2\mathbf{B} = \mathbf{B}$

$\Rightarrow \quad \mathbf{A}^2\mathbf{BB}^{-1} = \mathbf{BB}^{-1}$

$\Rightarrow \quad \mathbf{A}^2 = \mathbf{I}$

b

$\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

$\mathbf{BA} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

$\mathbf{AB} = \mathbf{BA} \Rightarrow b = c$

$d = a$

i.e. $a = d$ and $b = c$

**6 a** det $\mathbf{A} = \begin{vmatrix} 4 & p \\ -2 & -2 \end{vmatrix} = 4 \times (-2) - p \times (-2)$

$= -8 + 2p \neq 0$ for inverse to exist.

Hence $-8 + 2p \Rightarrow p \neq 4$

$\mathbf{A}^{-1} = \dfrac{1}{2p - 8}\begin{pmatrix} -2 & -p \\ 2 & 4 \end{pmatrix}$

$= \dfrac{1}{p - 4}\begin{pmatrix} -1 & \dfrac{-p}{2} \\ 1 & 2 \end{pmatrix}$

**6 b**

$$\mathbf{A}+\mathbf{A}^{-1}=\begin{pmatrix}4 & p\\ -2 & -2\end{pmatrix}+\begin{pmatrix}-\frac{1}{p-4} & \frac{p}{2p-8}\\ \frac{1}{p-4} & \frac{2}{p-4}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{4p-17}{p-4} & \frac{2p^2-9p}{2p-8}\\ \frac{9-2p}{p-4} & \frac{10-2p}{p-4}\end{pmatrix}=\begin{pmatrix}5 & \frac{9}{2}\\ -3 & -4\end{pmatrix}$$

Compare a corresponding element to find $p$

$$\frac{4p-17}{p-4}=5$$

$$4p-17=5(p-4)$$

$$4p-17=5p-20$$

$$5p-4p=-17+20$$

$$p=3$$

**7 a** $\det\mathbf{M}=\begin{vmatrix}k & -3\\ 4 & k+3\end{vmatrix}=k\times(k+3)-(-3)\times 4$

$$=k^2+3k+12$$

**b** Complete the square

$$k^2+3k+12=\left(k^2+3k+\frac{9}{4}\right)+12-\frac{9}{4}$$

$$=\left(k+\frac{3}{2}\right)^2+9.75\geq 9.75 \text{ for all } k$$

Hence, $\mathbf{M}$ is non-singular

**7 c**

$10\mathbf{M}^{-1} + \mathbf{M} = \mathbf{I}$

$\mathbf{M}^{-1} = \frac{1}{k^2 + 3k + 12}\begin{pmatrix} k+3 & 3 \\ -4 & k \end{pmatrix}$

$10\mathbf{M}^{-1} = \frac{10}{k^2 + 3k + 12}\begin{pmatrix} k+3 & 3 \\ -4 & k \end{pmatrix}$

$10\mathbf{M}^{-1} + \mathbf{M} = \frac{10}{k^2 + 3k + 12}\begin{pmatrix} k+3 & 3 \\ -4 & k \end{pmatrix} + \begin{pmatrix} k & -3 \\ 4 & k+3 \end{pmatrix}$

Top right element is $\frac{30}{k^2 + 3k + 12} - 3$

Compare with top right element of **I**

$\frac{30}{k^2 + 3k + 12} - 3 = 0$

$\frac{30}{k^2 + 3k + 12} = 3$

$3(k^2 + 3k + 12) = 30$

$3k^2 + 9k + 6 = 0$

$(3k + 6)(k + 1) = 0$

$k = -2$ inadmissible, does not give **I**

$k = -1$

**8 a** $\mathbf{M} = \begin{pmatrix} 2 & 3 \\ k & -1 \end{pmatrix}$

det $\mathbf{M} = -2 - 3k$

If the matrix has an inverse then det $\mathbf{M} \neq 0 \Rightarrow -2 - 3k \neq 0 \Rightarrow k \neq -\frac{2}{3}$

Therefore **M** has an inverse when $k < -\frac{2}{3}$ or $k > -\frac{2}{3}$

**b** $\mathbf{M}^{-1} = -\frac{1}{3k + 2}\begin{pmatrix} -1 & -3 \\ -k & 2 \end{pmatrix}$

**9 a** det $\mathbf{A} = \begin{vmatrix} a & 2 \\ 3 & 2a \end{vmatrix} = a \times 2a - 2 \times 3$

$= 2a^2 - 6$

$\mathbf{A}^{-1} = \frac{1}{2a^2 - 6}\begin{pmatrix} 2a & -2 \\ -3 & a \end{pmatrix}$

**b** $2a^2 - 6$

$2a^2 = 6$

$a^2 = 3$

$a = \sqrt{3}, a = -\sqrt{3}$

**Challenge**

Let $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

det $\mathbf{A} = ad - bc$ and det $\mathbf{B} = eh - fg$

det $\mathbf{A}$ det $\mathbf{B} = (ad - bc)(eh - fg)$

$= adeh - adfg - bceh + bcfg$

$\mathbf{AB}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

det $\mathbf{AB} = (ae + bg)(cf + dh) - (ce + dg)(af + bh)$

$= acef + adeh + bcfg + bdgh - acef - bceh - adfg - bdgh$

$= adeh + bcfg - bceh - adfg$

$=$ det $\mathbf{A}$ det $\mathbf{B}$